Positive paths in the linear symplectic group

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1 Introduction

A positive path in the linear symplectic group $\mathrm{Sp}(2n)$ is a smooth path which is everywhere tangent to the positive cone. These paths are generated by negative definite (time-dependent) quadratic Hamiltonian functions on Euclidean space. A special case are autonomous positive paths, which are generated by time-independent Hamiltonians, and which all lie in the set $\mathcal{U}$ of diagonalizable matrices with eigenvalues on the unit circle. However, as was shown by Krein, the eigenvalues of a general positive path can move off the unit circle. In this paper, we extend Krein's theory: we investigate the general behavior of positive paths which do not encounter the eigenvalue 1, showing, for example, that any such path can be extended to have endpoint with all eigenvalues on the circle. We also show that in the case $2n = 4$ there is a close relation between the index of a positive path and the regions of the symplectic group that such a path can cross. Our motivation for studying these paths came from a geometric squeezing problem [16] in symplectic topology. However, they are also of interest in relation to the stability of periodic Hamiltonian systems [9] and in the theory of geodesics in Riemannian geometry [4].

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Main results

We consider $\mathbf{R}^{2n}$ equipped with the standard (linear) symplectic form

$$\omega(X, Y) = Y^T J X,$$

where J is multiplication by i in $\mathbf{R}^{2n} \cong \mathbf{C}^n$. Thus if $e_1, \ldots, e_{2n}$ form the standard basis $Je_{2i-1} = e_{2i}$. The Lie algebra $\mathbf{sp}(2n)$ of $G = \mathrm{Sp}(2n, \mathbf{R})$ is the set of all matrices which satisfy the equation

$$A^T J + J A = 0.$$

Hence $A \in \mathbf{sp}(2n)$ if and only if JA is symmetric. The symplectic gradient (or Hamiltonian vector field) X_H of a function $H : \mathbf{R}^{2n} \to \mathbf{R}$ is defined by the identity

$$\omega(X_H, Y) = Y^T J X_H = dH(Y) \quad \text{for all } Y \in \mathbf{R}^{2n}.$$

Given a symmetric matrix P, consider the associated function

$$\mathcal{Q}(x) = -\frac{1}{2} x^T P x$$

on $\mathbf{R}^{2n}$. Then the calculation

$$d\mathcal{Q}_x(Y) = -Y^T P x = Y^T J(JPx)$$

shows that the symplectic gradient of $\mathcal{Q}$ is the vector field

$$X_{\mathcal{Q}}(x) = JP(x), \quad \text{at} \quad x \in \mathbf{R}^{2n},$$

which integrates to the linear flow $x \mapsto e^{JPt} x$. We call these paths $\{e^{JPt}\}_{t \in [0,1]}$ **autonomous**, since they are generated by autonomous (i.e. time-independent) Hamiltonians.

We are interested in paths $A_t \in G$ which are **positive** in the sense that their tangent vector at time t has the form $JP_t A_t$ for some symmetric, positive definite matrix P_t. Thus they are generated by a family of time-dependent quadratic Hamiltonians[1] of the form $\mathcal{Q}_t = -\frac{1}{2} x^T P_t x$. These paths are everywhere tangent to the **positive cone field** $\mathcal{P}$ on G given by

$$\cup_{A \in G} \mathcal{P}_A = \cup_{A \in G} \{JPA : P = P^T, P > 0\} \subset \cup_A T_A G = TG.$$

In particular, an autonomous path e^{JPt} is positive if P is.

Note that an autonomous path e^{JPt} always belongs to the set $\mathcal{U}$ of elements in G which are diagonalizable (over $\mathbf{C}$) and have spectrum on the unit circle. The following result shows that this is not true for all positive paths.

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[1] It is somewhat unfortunate that our sign conventions imply that $\mathcal{Q}_t$ is actually negative definite.

Proposition 1.1 *Any two elements in* G *may be joined by a positive path.*

This is a special case of a general result in control theory known as Lobry's theorem: see Lobry [17], Sussmann [19] and Grasse–Sussmann [11]. We give a proof here in §4.2 as a byproduct of our other results.

The structure of positive paths $A_{t \in [0,1]}$ in G was studied by Krein in a series of papers during the first half of the 50's. Later Gelfand–Lidskii [9] took up the study of the topological properties of general paths in connection with the stability theory of periodic linear flows. Almost simultaneously, and surely independently, Bott [3, 4, 5] studied positive flows in the complexified linear group in connection with the geometry of closed geodesics. As a result of these works, the structure of positive paths is well-understood provided that the spectrum of A_t remains on the unit circle. However, we need to investigate their behavior outside this circle. Surprisingly, a thorough study of the behavior of these paths outside the unit circle does not seem to exist. The aim of this article is to undertake such a study, at least in the case of generic positive paths, that is to say those paths which meet only the codimension 0 and codimension 1 strata of the matrix group. For the convenience of the reader we will develop the theory from scratch, even though some of our first results are well-known.

Our main result is motivated by the geometric application in [16]. Before stating this result, we introduce some notation. As above, $\mathcal{U}$ denotes the set of elements in G which are diagonalizable and have spectrum on the unit circle (unless indicated to the contrary, "diagonalizable" is considered over **C**). Further,

$$\mathcal{S}_1 = \{A \in G : \det(A - \mathbb{1}) = 0\}$$

is the set of elements with at least one eigenvalue equal to 1. A path $\gamma = A_{t \in [0,1]}$ in G which starts at the identity $A_0 = \mathbb{1}$ and is such that $A_t \in G - \mathcal{S}_1$ for all $t > 0$ will be called **short**. Equivalently, it has Conley-Zehnder index equal to 0.[2] Moreover, an element $A \in G$ is called **generic** if all its eigenvalues (real or complex) have multiplicity 1. In the next theorem, all paths begin at $\mathbb{1}$.

Theorem 1.2 (i) *An element of* $G - \mathcal{S}_1$ *is the endpoint of a short positive path if and only if it has an even number of real eigenvalues* λ *with* $\lambda > 1$.

(ii) *Any short positive path may be extended to a short positive path with endpoint in* $\mathcal{U}$.

(iii) *The space of short positive paths with endpoint in* $\mathcal{U}$ *is path-connected.*

We can interpret these results in terms of a "conical" subRiemannian geometry. SubRiemannian geometry is usually a study of paths in a manifold M which are everywhere tangent to some distribution ξ in the tangent bundle

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[2] The Conley–Zehnder index (or Maslov index) of a path measures how many times it goes through the eigenvalue 1 and characterizes its homotopy class: see Gelfand–Lidskii [9], Ekeland [7, 8], Robbin–Salamon /citeROSA1.

TM. For example, there is recent interest in the case when ξ is a contact structure: see [12]. One could generalize this to the consideration of paths which are everywhere tangent to some fixed convex conical neighborhood $\mathcal{N}_\varepsilon(\xi)$ of ξ. The geometry of positive paths more or less fits into this context: one just has to replace the distribution ξ by a distribution of rays. Thus we can consider the positive cone to be a neighborhood of the ray generated by the right invariant vector field $JA \in T_A\mathrm{G}$ corresponding to the element J in the Lie algebra. Proposition 1.1 shows that G remains path-connected in this geometry. Next, one might try to understand what happens to the fundamental group. More precisely, let us define the **positive fundamental group** $\pi_{1,\mathrm{pos}}\mathrm{G}$ to be the semigroup generated by positive loops based at $\mathbb{1}$, where two such loops are considered equivalent if they may be joined by a smooth family of positive loops. It would be interesting to calculate this semigroup. In particular it is not at present clear whether or not the obvious map

$$\pi_{1,\mathrm{pos}}\mathrm{G} \;\to\; \pi_1(\mathrm{G})$$

is injective in general. (When $n = 1$ this is an easy consequence of Proposition 2.4 and Lemma 3.1.)

The proof of Theorem 1.2 is based on studying the relation between the positive cone and the fibers of the projection $\pi : \mathrm{G} \to \mathcal{C}onj$, where $\mathcal{C}onj$ is the space of conjugacy classes of elements in G with the quotient (non-Hausdorff) topology. The space $\mathcal{C}onj$ has a natural stratification coming from the stratification of G. Recall that a symplectic matrix A is similar to its transpose inverse $(A^T)^{-1}$ (in fact, $(A^T)^{-1} = -JAJ$). Thus its eigenvalues occur either in pairs $\lambda, \bar{\lambda} \in S^1$ and $\lambda, 1/\lambda \in \mathbf{R} - 0$, or in complex quadruplets $\lambda, \bar{\lambda}, 1/\lambda, 1/\bar{\lambda}$. In particular, the eigenvalues ± 1 always occur with even multiplicity. Roughly speaking, the open strata of G consist of generic elements (ie diagonalizable elements with all eigenvalues of multiplicity 1), and the codimension 1 strata consist of non-diagonalizable elements which have one pair of eigenvalues of multiplicity 2 lying either on $\mathbf{R} - \{\pm 1, 0\}$ or on $S^1 - \{\pm 1\}$, or which have one pair of eigenvalues equal to ± 1: see [1] and §3 below. Note that a conjugacy class which lies in an open stratum is a submanifold of $\mathrm{GL}(2n, \mathbf{R})$ of codimension n since each pair of distinct eigenvalues (in $\mathbf{R} \cup S^1$) has one degree of freedom, and each quadruplet has two.

We write G_0 for the set of all generic elements in G, G_1 for the union of all codimension 1 strata, and similarly $\mathcal{C}onj_0, \mathcal{C}onj_1$ for their projections in $\mathcal{C}onj$. We shall consider in detail only generic paths in $\mathcal{C}onj$. By definition, these intersect all lower strata transversally. Hence the elements of these paths lie in $\mathcal{C}onj_0$ for all except the finite number of times at which they cross $\mathcal{C}onj_1$. Note that the codimension of a stratum in $\mathcal{C}onj$ always refer to the codimension of its lift in G.

The main ingredient of the proof of the theorem is to characterize the (generic) paths in $\mathcal{C}onj$ which lift to positive paths in G. We shall see in Proposition 3.6 that the only significant restriction on the path comes from the way

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that eigenvalues enter, move around and leave S^1. We will not state the general result here since it needs a certain amount of notation. However some of the essential features are present in the case $n = 1$ which is much easier to describe. In this case $\mathcal{C}onj$ is the union of the circle with the intervals $(-\infty, -1] \cup [1, \infty)$ with the usual topology, except that the points ± 1 are each tripled. To see this, first consider matrices with an eigenvalue pair $\lambda, \bar\lambda \in S^1$. It is possible to distinguish between these eigenvalues and hence to label the pair with an element in S^1. In higher dimensions this is accomplished by the *splitting number* described in §2. Here it suffices to note that the rotations through angles θ and $2\pi - \theta$ are not conjugate in $\mathrm{SL}(2, \mathbf{R})$, the only conjugating matrices being reflections. We also claim that the point 1 occurs with three flavors: plain $\mathbb{1}$ (the conjugacy class of the identity) which has codimension 3, and two nilpotent classes $\mathbb{1}^{\pm} \in \mathcal{C}onj_1$. A similar statement holds at -1. As explained in Lemma 3.1, a positive path has to project to a path in $\mathcal{C}onj$ which goes round the circle anticlockwise, but it can move along the real axes in either direction. Moreover, the projection of a generic positive path has to leave the circle at $(-\mathbb{1})^-$ or $\mathbb{1}^-$ and then, after wandering around $(-\infty, -1) \cup (1, \infty)$ for a while, enter the circle again at $(-\mathbb{1})^+$ or $\mathbb{1}^+$ and continue on its way.

The next step is to carry out a similar analysis in the case $n = 2$. We shall see that the restrictions in the behavior of positive paths in this case all stem from Krein's lemma which says that simple eigenvalues on S^1 with positive splitting number must flow anti-clockwise round the circle. This gives rise to restrictions in the way in which four eigenvalues on the unit circle can move to a quadruplet outside the unit circle (the Krein–Bott bifurcation). As a consequence, we show in Proposition 5.4 that there are restrictions on the regions in $\mathcal{C}onj$ which can be visited by a generic positive path of bounded Conley-Zehnder index.

In the last section of the paper, we will present a brief survey of the use of positive paths in stability theory, in Riemannian geometry and in Hofer geometry, and will give some new applications.

We wish to thank Nancy Hingston for describing to us her ideas about perambulations in the symplectic group, which were a direct inspiration for our work, and Hector Sussmann for explaining to us some basic results in control theory.

2 Basic facts

2.1 General results on positive paths

First, two elementary lemmas. A piecewise smooth path is said to be piecewise positive if the tangent vectors along each smooth segment (including those at the endpoints) are in the positive cone.

Lemma 2.1 (i) *The set of positive paths is open in the C^1-topology.*

(ii) *Any piecewise positive path may be C^0-approximated by a positive path.*

(iii) *Let $\{A_t\}, \{B_t\}$ be two positive paths with the same initial point $A_0 = B_0$. Then, given $\varepsilon > 0$, there is $\delta > 0$ and a C^0-small deformation of $\{A_t\}$ to a positive path $\{A_t'\}$ which coincides with $\{B_t\}$ for $t \in [0, \delta]$ and with $\{A_t\}$ for $t > \varepsilon$.*

Proof: The first statement follows from the openness of the positive cone, the second from its convexity and the third from a combination of these two facts. □

Note that statement (ii) above is a special case of a much more general result in control theory: see, for example, Sussmann [19] and Grasse–Sussmann [11].

Lemma 2.2 (i) *Any positive path can be extended to a positive path which ends at a generic element.*

(ii) *Any positive path $\{A_t\}$ starting at $A_0 = 1\!\!1$ can be perturbed fixing A_0 so that $\{A_t\}$ is generic. Moreover, if A_1 is generic we may fix that also during the perturbation.*

Proof: This follows immediately from openness. □

We continue with some simple remarks about the relation between positive paths in G and their projections (also called positive paths) in $\mathcal{C}onj$.

Lemma 2.3 (i) *The positive cone is invariant under conjugation.*

(ii) *The conjugate $\{X^{-1}A_tX\}$ of any positive path $\{A_t\}$ is positive.*

(iii) *There is a positive path from $1\!\!1$ to A in G if and only if there is a positive path in $\mathcal{C}onj$ from $1\!\!1$ to $\pi(A)$.*

Proof: (i) The positive vector JPA at A is taken by conjugation to the positive vector $X^{-1}JPAX = X^{-1}JPXX^{-1}AX = J(X^TPX)B$ at $B = X^{-1}AX$.
(ii) follows immediately from (i).
(iii) Since the "only if" statement is obvious, we consider the "if" statement. By definition, a positive path from $\pi(1\!\!1)$ to $\pi(A)$ lifts to a positive path from $1\!\!1$ to $Y^{-1}AY$ for some $Y \in G$. Now conjugate this path by Y^{-1} and use (ii). □

Warning Given a positive path $\{A_t\}$ it is *not* true that all paths of the form $\{X_t^{-1}A_tX_t\}$ are positive. One can easily construct counterexamples in $\mathrm{SL}(2, \mathbf{R})$ using the methods of §3.

Proposition 2.4 *Let $\{A_t\}$ in G be a generic positive path joining two generic points A_0, A_1. Then the set of positive paths in G which lift $\gamma = \{\pi(A_t)\}_{t\in[0,1]} \subset \mathcal{C}onj$ is path-connected. Moreover, the set of these paths with one fixed endpoint is also path-connected. Finally, if $A_0 = 1\!\!1$, the set of these paths with both endpoints fixed is path-connected.*

Proof: Let $\{B_t\}$ be another path which lifts $\gamma = \{\pi(A_t)\}$. By genericity we may suppose that $\{A_t\}$ and $\{B_t\}$ are disjoint embedded paths. Moreover, when $n \geq 2$, the projection in $\mathcal{C}onj$ of the positive cone at any point $X \in G_0 \cup G_1$ of a generic path is always at least 2-dimensional (see Proposition 3.6). Hence we may assume that the paths $\{A_t\}$ and $\{B_t\}$ are never tangent to the fiber of the projection $G \to \mathcal{C}onj$. Our first aim is to define positive piecewise smooth vector fields ξ_A, ξ_B that are tangent to $\pi^{-1}(\gamma)$ and which extend the tangent vector fields to the paths $\{A_t\}, \{B_t\}$. To make this easier we first normalise the path B_t near the finite number of times $t_1 < \ldots < t_k$ at which A_t crosses a codimension 1 stratum. If T is one of these times t_i, there is a matrix X_T such that $B_T = X_T^{-1} A_T X_T$, and by Lemma 2.1 (iii) we may suppose that $B_t = X_T^{-1} A_t X_T$ for t near T.

On the other hand, since the set of tangent vectors $Y \in TG$ which projects to the tangent vector $\pi_*(\dot{A}_T)$ is an affine subbundle of the restriction $TG\mid_{\pi^{-1}(\pi(A_T))}$, we can find a section of this subbundle whose value at A_T coincides with $\dot{A}_T$. This defines ξ_A over neighbourhoods of all times t_i, which we choose small enough so that ξ_A is positive. We then define ξ_B over each such neighbourhood by the adjoint map $X_T^{-1} \xi_A X_T$.

At all other times A_t is generic and the map $t \mapsto \pi(A_t)$ is an immersion. Therefore, the set

$$Y_i = \{\pi^{-1}(\pi(A_t)) : t_i \leq t \leq t_{i+1}\}$$

is a submanifold. We now extend ξ_A, ξ_B to the whole of Y_i by choosing sections of the intersection of the positive cone field with TY_i which project to the tangent vector field of γ. This is possible because this intersection is an open convex cone. Observe that π projects all integral curves of ξ_A and ξ_B to γ. Indeed all integral curves of the vector field $s\xi_A + (1-s)\xi_B$ (for $s \in [01]$) are positive paths which project to γ. Clearly there is a path of such curves joining $\{A_t\}$ to $\{B_t\}$.

It remains to check the statement about the endpoints. We will assume, again using Lemma 2.1, that if $\{A_t\}$ and $\{B_t\}$ have the same endpoint they agree for t near that endpoint. Then, if $B_0 = A_0$ the above construction gives a family of paths starting at A_0. Therefore the result for fixed initial endpoint is obvious. To get fixed final endpoint one can use a similar argument applied to the reversed (negative) paths $\{A_{1-t}\}, \{B_{1-t}\}$. Finally consider the case of two fixed endpoints, with $A_0 = B_0 = \mathbb{1}$. The endpoints A_1^λ of the family of paths constructed above are all conjugate to A_1 and equal A_1 at $\lambda = 0, 1$. Because A_1 is generic, there is a smooth family of elements $X_\lambda \in G$ with $X_0, X_1 = \mathbb{1}$ such that $X_\lambda^{-1} A_1^\lambda X_\lambda = A_1$. The desired family of paths is then $\{X_\lambda^{-1} A_t^\lambda X_\lambda\}_{t \in [0,1]}$: these are positive by Lemma 2.3. □

2.2 Splitting numbers

This section summarizes what is known as Krein theory, that is the theory of positive paths in $\mathcal{U}$. The theory is nicely described by Ekeland in [7, 8] where it

is used to develop an index theory for closed orbits. Interestingly enough, this analysis is very closely related to work by Ustilovsky [20] on conjugate points on geodesics in Hofer geometry on the group of Hamiltonian symplectomorphisms.

It will be convenient to work over $\mathbf{C}$ rather than $\mathbf{R}$. We extend J and ω to $\mathbf{C}^{2n} = \mathbf{R}^{2n} \otimes \mathbf{C}$ by complex linearity, so that $\omega(v, w) = w^T J v$ is complex bilinear. Let $\langle v, w \rangle$ denote the standard Hermitian inner product on $\mathbf{C}^n$, namely

$$\langle v, w \rangle = \bar{w}^T v = \omega(Jv, \bar{w}).$$

We will also use the form β given in our notations by

$$\beta(v, w) = -i\omega(v, \bar{w}) = -i\langle Jv, w \rangle = -i\bar{w}^T Jv.$$

Lemma 2.5 (i) β *is a nondegenerate Hermitian symmetric form.*

(ii) $i\beta(v, Jv) = \langle v, v \rangle$.

(iii) *An element $A \in \mathrm{GL}(2n, \mathbf{C})$ belongs to G if and only if it preserves β and is real, i.e.*
$$\overline{Av} = A\bar{v}, \quad v \in \mathbf{C}^{2n}.$$

(iv) *The invariant subspaces E_λ, E_μ corresponding to eigenvalues λ, μ are β-orthogonal unless $\lambda\bar{\mu} = 1$.*

(v) *If v is an eigenvector with eigenvalue $\lambda \in S^1$ of multiplicity 1 then $\beta(v, v) \in \mathbf{R} - \{0\}$.*

Proof: β is Hermitian because

$$\overline{\beta(w, v)} = \overline{-i\omega(w, \bar{v})} = i\omega(\bar{w}, v) = -i\omega(v, \bar{w}) = \beta(v, w).$$

Statements (ii) and (iii) are obvious. To prove (iv), consider $A \in \mathrm{G} = \mathrm{Sp}(2n, \mathbf{R})$ with eigenvalues λ, μ and corresponding eigenvectors v, w and observe that

$$\beta(v, w) = \beta(Av, Aw) = \beta(\lambda v, \mu w) = \lambda\bar{\mu}\beta(v, w).$$

The proof when v, w belong to the invariant subspace but are not eigenvectors is similar (see Ekeland [8]). To prove (v), suppose $\lambda \in S^1$ is an eigenvalue of multiplicity 1 with eigenvector v. Then $\beta(v, v) \neq 0$, since β is non-degenerate and $\beta(v, w) = 0$ whenever w belongs to any other eigenspace of A. Further, $\beta(v, v) \in \mathbf{R}$ by (i). $\qquad\square$

It follows from (v) that simple eigenvalues λ on S^1 may be labelled with a number $\sigma(\lambda) = \pm 1$ called the **splitting number** chosen so that

$$\beta(v, v) \in \sigma(\lambda)\mathbf{R}^+.$$

For example, when $n = 1$, the rotation matrix

$$\rho(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

has eigenvectors

$$v^+ = \begin{pmatrix} 1 \\ -i \end{pmatrix}, \quad v^- = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

corresponding to the eigenvectors e^{it}, e^{-it}, and it is easy to see that e^{it} has splitting number $+1$, while e^{-it} has splitting number -1.

More generally, if $\lambda \in S^1$ has multiplicity > 1 we define $\sigma(\lambda)$ to be the signature of the form β on the corresponding invariant subspace. Note that $\sigma(\lambda) = -\sigma(\bar{\lambda})$. Hence the splitting number of ± 1 is always 0. It is not hard to check that the conjugacy class of an element in $\mathcal{U}$ is completely described by its spectrum together with the corresponding set of splitting numbers. The following result is central to our argument.

Lemma 2.6 (Krein) *Under a positive flow simple eigenvalues on S^1 labelled with $+1$ must move anti-clockwise and those labelled with -1 must move clockwise.*

Proof: We repeat the proof from Ekeland's book [8] for the convenience of the reader. For all t in some neighborhood of t_0, let $e^{i\theta_t}$ be a simple eigenvalue of A_t with corresponding eigenvector $x_t \in \mathbf{C}$. For simplicity when $t = t_0$ we use the subscript 0 instead of t_0 (writing A_0, x_0 etc), and will denote the derivative of A_t at t_0 by JPA_0, where P is positive definite. It is easy to check that for any x

$$\langle A_t x, J x_t \rangle = \langle x, A_t^T J x_t \rangle = \langle x, J A_t^{-1} x_t \rangle = \langle e^{i\theta_t} x, J x_t \rangle.$$

Applying this with $x = x_0$ and differentiating at $t = t_0$, we find that

$$\langle JPA_0 x_0, J x_0 \rangle = \langle i \frac{d\theta_t}{dt} e^{i\theta_0} x_0, J x_0 \rangle \bigg|_{t=t_0},$$

from which it readily follows that

$$\frac{d\theta_t}{dt} \bigg|_{t=t_0} = \frac{\langle Px_0, x_0 \rangle}{\beta(x_0, x_0)}.$$

Since the right hand side has the same sign as $\beta(x_0, x_0)$, the result follows. $\square$

Observe, however, that not all flows whose eigenvalues move in this way are positive. Further, this result shows that there may not be a short positive path between an arbitrary pair of elements in a given conjugacy class, even if one allows the path to leave the conjugacy class.

The next lemma shows that if a (simple) eigenvalue leaves the circle it must do so at a point with splitting number 0. This observation highlights the importance of the splitting number. It is not relevant to the case $n = 1$ of course, but is a cornerstone of the argument in higher dimensions.

Lemma 2.7 *Let A_t be any path in G and $\lambda(t) \in \operatorname{Spec} A_t$ a continuous family of eigenvalues such that*

$$\lambda(t) \in S^1, \quad for \ \ t \leq T,$$
$$\lambda(t) \notin S^1, \quad for \ \ t > T.$$

Suppose also that $\lambda(t)$ has multiplicity 1 when $t > T$ and multiplicity 2 at $t = T$. Then

$$\sigma(\lambda_T) = 0.$$

Proof: For $t \geq T$ let $V_t \subset \mathbf{C}^{2n}$ be the space generated by the eigenspaces $E_{\lambda(t)}, E_{1/(\bar\lambda(t))}$. By hypothesis, this is 2-dimensional for each such t, and it clearly varies continuously with t. As above, β is non-degenerate on each V_t. Therefore the splitting number can be $2, 0$ or -2 and will be 0 if and only if there are non-zero v such that $\beta(v, v) = 0$. But when $t > T$, $\beta(v, v) = 0$ on the eigenvectors in V_t. Therefore, β has signature 0 when $t > T$ and so must also have signature 0 on V_T. $\qquad \square$

3 Conjugacy classes and the positive cone

3.1 The case $n = 1$

We will write $\mathcal{O}_{\mathcal{R}}^{\pm}$ for the set of all elements in $\mathrm{SL}(2, \mathbf{R})$ with eigenvalues in $\mathbf{R}^{\pm} - \{\pm 1\}$, and will also divide $\mathcal{U} - \{\pm \mathbb{1}, \pm \mathbb{1}^{\pm}\}$ into two sets $\mathcal{O}_{\mathcal{U}}^{\pm}$, distinguished by the splitting number of the eigenvalue λ in the upper half-plane

$$\mathbf{H} = \{z \in \mathbf{C} : \Im\, z > 0\}.$$

Thus, for elements of $\mathcal{O}_{\mathcal{U}}^+$, the eigenvalue in $\mathbf{H}$ has splitting number $+1$, and, by considering the rotations

$$\rho(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix},$$

it is easy to check that

$$\mathcal{O}_{\mathcal{U}}^+ = \{A : \operatorname{tr} A < 2,\ A_{21} > 0\}, \quad \mathcal{O}_{\mathcal{U}}^- = \{A : \operatorname{tr} A < 2,\ A_{21} < 0\}.$$

These sets project to the open strata in $\mathcal{C}onj$.

We now work out in detail the structure of conjugacy classes near the element $-\mathbb{1}$. The structure at $\mathbb{1}$ is similar, but will not be so important to us since we are interested in paths which avoid the eigenvalue 1.

It is not hard to check that there are 3 conjugacy classes with spectrum $\{-1\}$, the diagonalizable class consisting only of $\{-\mathbb{1}\}$, and the classes $\mathcal{N}^{\pm}$ containing the nilpotent elements

$$\begin{pmatrix} -1 & 0 \\ \mp 1 & -1 \end{pmatrix}.$$

Again, the classes $\mathcal{N}^\pm$ may be distinguished by the sign of the 21-entry. Moreover, if we write the matrices near $-\mathbb{1}$ as

$$A = \begin{pmatrix} -1+x & y \\ z & \frac{1+yz}{-1+x} \end{pmatrix},$$

the elements in $\mathcal{N}^- \cup \{-\mathbb{1}\} \cup \mathcal{N}^+$ are those with trace $= -2$ and so form the boundary of the cone $x^2 + yz = 0$. The interior of the cone is the set where $x^2 < -yz$, and has two components

$$\mathcal{U}^+ = \{y < 0,\ z > 0\}, \quad \mathcal{U}^- = \{y > 0,\ z < 0\}.$$

The rest of the space consists of points in $\mathcal{O}_\mathcal{R}^-$ with $x^2 > -yz$. Observe that the labelling is chosen so that $\mathcal{N}^-$ is in the closure of $\mathcal{O}_\mathcal{U}^+$. (This class $\mathcal{N}^-$ was labelled by $-$ rather than $+$ because, as we shall see, it is the place where positive paths leave $\mathcal{U}$.)

The next lemma describes the structure of short positive paths in $\mathrm{SL}(2, \mathbf{R})$.

Lemma 3.1 *Let $A_{t \in [0,1]}$ be a short positive path in $\mathrm{SL}(2, \mathbf{R})$, and for each $t > 0$, let*

$$\lambda_t = r(t) e^{i\theta_t}, \quad \theta_t \in (0, 2\pi), \ |r(t)| \geq 1,$$

be an eigenvalue of A_t, chosen so that it has splitting number $+1$, when $\theta \neq \pi$. Then:

(i) *there is $\varepsilon > 0$ such that $A_t \in \mathcal{O}_\mathcal{U}^+$ for $t \in [0, \varepsilon]$;*

(ii) *the function θ_t is increasing, and is strictly increasing except perhaps at the point $\theta_t = \pi$;*

(iii) *if $A_T = -\mathbb{1}$ for any T then A_t remains in $\mathcal{U}$ for $t > T$;*

(iv) *if A_t enters $\mathcal{O}_\mathcal{R}^-$ it does so through a point of $\mathcal{N}^-$ and if it reenters $\mathcal{U}$ it must go through $\mathcal{N}^+$ to $\mathcal{O}_\mathcal{U}^-$.*

(v) *$A_t \notin \mathcal{O}_\mathcal{R}^+$ for $t > 0$.*

Proof: We first claim that at every point N of $\mathcal{N}^-$ all vectors in the positive tangent cone at N point into $\mathcal{O}_\mathcal{R}^-$. For if not, by openness (Lemma 2.1(i)), there would be a positive vector pointing into $\mathcal{O}_\mathcal{U}^+$ and hence a positive path along which a positive eigenvalue would move clockwise, in contradiction to Krein's lemma. Since conjugation by an orthogonal reflection interchanges $\mathcal{N}^+$ and $\mathcal{N}^-$ and positive and negative, it follows that the positive tangent cone points in $\mathcal{O}_\mathcal{U}^-$ at all points of $\mathcal{N}^+$. Since every neighbourhood of $-\mathbb{1}$ contains points in $\mathcal{N}^\pm$, it follows from openness that positive paths starting at $-\mathbb{1}$ must remain in $\mathcal{U}$. Since a similar result holds for negative paths, there is no short positive path which starts with eigenvalues on $\mathbf{R}^-$ and ends at $-\mathbb{1}$. This proves (iii) and (iv). A similar argument proves (i) and (v). Statement (ii) is a direct consequence of Lemma 2.6). $\square$

Remark 3.2 (i) The above proof shows that the structure of these positive paths in $\mathrm{SL}(2, \mathbf{R})$ is determined by Krein's lemma and the topology of the conjugacy classes. However the result may also be proved by direct computation. For example, if

$$N = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} \in \mathcal{N}^+$$

and P is symmetric and positive definite, then

$$\frac{d}{dt}\big|_{t=0} \, \mathrm{tr}\,(e^{tJP}N) = \mathrm{tr}\,(JPN) > 0.$$

Hence the trace lies in $(-2, 2)$ for $t > 0$, which means that every positive path starting at N moves into $\mathcal{U}^-$. By invariance under conjugacy, this has be to true for every element in $\mathcal{N}^+$.

(ii) It is implicit in the above proof that under a positive flow in $\mathrm{SL}(2, \mathbf{R})$ eigenvalues on $\mathbf{R}$ can flow in either direction. To see this explicitly, let A be the diagonal matrix $\mathrm{diag}(\lambda, 1/\lambda)$ (where $|\lambda| > 1$) and consider the positive matrices

$$P_1 = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then the derivative of the trace (and hence the flow direction of λ) has different signs along the flows $t \mapsto e^{JP_i t}A, i = 1, 2$.

3.2 The case $n = 2$

This case contains all the complications of the general case. We will see that the behavior of eigenvalues along positive paths as they enter or leave the circle at a value in $S^1 - \{\pm 1\}$ is much the same as the behavior that we observed in the case $n = 1$ at the triple points ± 1 in $\mathcal{C}onj$. When $n = 2$ it is also possible for two pairs of real eigenvalues to come together and then leave the real axis. However, we shall see that the positivity of the path imposes no essential restiction here. One indication of this is that there is no relevant notion of splitting number when λ is not on the unit circle.

We first consider the structure of elements $A \in \mathrm{Sp}(4, \mathbf{R})$. As remarked above, the eigenvalues of A consist either of pairs $\lambda, \bar{\lambda} \in S^1$ and $\lambda, 1/\lambda \in \mathbf{R} - \{0\}$, or of quadruplets $\lambda, \bar{\lambda}, 1/\lambda, 1/\bar{\lambda}$ where $\lambda \notin S^1 \cup \mathbf{R}$. We will label these pairs and quadruplets by the element λ with $|\lambda| \geq 1$ and $\Im\lambda \geq 0$.

A generic element of G lies in one of the following open regions:

(i) $\mathcal{O}_\mathcal{C}$, consisting of all matrices whose spectrum is a quadruplet in $\mathbf{C} - (\mathbf{R} \cup S^1)$;

(ii) $\mathcal{O}_\mathcal{U}$, consisting of all matrices whose eigenvalues lie in $S^1 - \{\pm 1\}$ and either all have multiplicity 1 or have multiplicity 2 and non-zero splitting numbers[3];

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[3] By Lemma 2.7 the latter elements must be diagonalizable.

(iii) $\mathcal{O}_\mathcal{R}$, consisting of all matrices whose eigenvalues have multiplicity 1 and lie in $\mathbf{R} - \{0, \pm 1\}$ (divided into two components: $\mathcal{O}_\mathcal{R}^\pm$);

(iv) $\mathcal{O}_{\mathcal{U},\mathcal{R}}$, consisting of all matrices with 4 distinct eigenvalues, one pair on $S^1 - \{\pm 1\}$ and the other on $\mathbf{R} - \{0, \pm 1\}$.

Note that the region $\mathcal{O}_\mathcal{C}$ is connected, but the others are not.

Lemma 3.3 *The codimension* 1 *part of the boundaries of the above regions are:*

(v) $\mathcal{B}_\mathcal{U}$, *consisting of all non-diagonalizable matrices whose spectrum consists of a pair of conjugate points in $S^1 - \{\pm 1\}$ each of multiplicity 2 and splitting number 0.*

(vi) $\mathcal{B}_\mathcal{R}$, *consisting of all non-diagonalizable matrices whose spectrum is a pair of distinct points $\lambda, 1/\lambda \in \mathbf{R} - \{\pm 1, 0\}$ each of multiplicity 2.*

(vii) $\mathcal{B}_{\mathcal{U},\mathcal{R}}$, *consisting of all non-diagonalizable matrices with spectrum $\{\pm 1, \pm 1, \lambda, \bar{\lambda}\}$, with $\lambda \in S^1 - \{\pm 1\}$.*

(viii) $\mathcal{B}_{\mathcal{R},\mathcal{U}}$, *consisting of all non-diagonalizable matrices with spectrum $\{\pm 1, \pm 1, \lambda, 1/\lambda\}$, with $\lambda \in \mathbf{R} - \{\pm 1, 0\}$.*

Proof: Use Lemma 2.7. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The next step is to describe the conjugacy classes of elements in the above parts of G. It follows easily from Lemma 2.7 that the conjugacy class of an element $A \in \mathcal{O}_\mathcal{U} \cup \mathcal{O}_\mathcal{R} \cup \mathcal{O}_{\mathcal{U},\mathcal{R}}$ is determined by its spectrum and the splitting numbers of those eigenvalues lying on S^1. The next lemma deals with the other cases.

Lemma 3.4 (i) *The conjugacy class of $A \in \mathcal{O}_\mathcal{C}$ is entirely determined by its eigenvalue $\lambda \in \mathbf{H}$ with $|\lambda| > 1$.*

(ii) *There are two types of conjugacy class in $\mathcal{B}_\mathcal{U}$. More precisely, for each eigenvalue $\lambda \in S^1 \cap \mathbf{H}$ of multiplicity 2 and splitting number 0, there are two conjugacy classes of non-diagonalizable matrices, namely the two nilpotent classes $\mathcal{N}_\lambda^\pm$.*

(iii) *For each $\lambda \in (-\infty, -1) \cup (1, \infty)$, there is one type of conjugacy class in $\mathcal{B}_\mathcal{R}$.*

(iv) *A matrix in $\mathcal{B}_{\mathcal{U},\mathcal{R}}$ or $\mathcal{B}_{\mathcal{R},\mathcal{U}}$ is conjugate to a matrix which respects the splitting $\mathbf{R}^4 = \mathbf{R}^2 \oplus \mathbf{R}^2$. Thus the conjugacy classes of $\mathcal{B}_{\mathcal{U},\mathcal{R}}$ are determined by $\lambda \in (0, \pi)$, its splitting number, and the conjugacy class $\mathcal{N}^-$ or $\mathcal{N}^+$ of the case $n = 1$. Those of $\mathcal{B}_{\mathcal{R},\mathcal{U}}$ are determined by $\lambda \in (-\infty, -1) \cup (1, \infty)$ and the conjugacy class $\mathcal{N}^-$ or $\mathcal{N}^+$ of the case $n = 1$.*

Proof: First consider an element $A \in \mathcal{O}_\mathcal{C}$. By Lemma 2.5(iv), the subspace V spanned by the eigenvectors v, w with eigenvalues $\lambda, 1/\bar{\lambda}$ is β-orthogonal to its complex conjugate $\bar{V}$, because this is the span of the eigenvectors $\bar{v}, \bar{w}$ with eigenvalues $\bar{\lambda}, 1/\lambda$. By the same Lemma, $\beta(v, v) = \beta(w, w) = 0$. Hence, because it is non-degenerate, the form β is non-zero on (v, w). Thus β has zero signature on the subspaces $V, \bar{V}$. Conversely, suppose given a 2-dimensional subspace V of $\mathbf{C}^4$ which is β-orthogonal to its complex conjugate $\bar{V}$. Let B be any complex linear β-preserving automorphism of V and extend B to $\bar{V}$ by complex conjugation. Then B is an element of $\mathrm{Sp}(4, \mathbf{R})$ because it is real and preserves β. In particular, if β has signature 0 on V, the subspace V is spanned by vectors v, w such that $\beta(v, v) = \beta(w, w) = 0$. Then $\beta(v, w) \neq 0$ since β is non-degenerate, and if we define $B = B(V, \lambda)$ by setting:

$$Bv = \lambda v, \quad Bw = (1/\bar{\lambda})w,$$

B preserves β because

$$\beta(Bv, Bw) = \beta(\lambda v, 1/\bar{\lambda}w) = \beta(v, w).$$

The corresponding element of $\mathrm{Sp}(4, \mathbf{R})$ has spectrum $\lambda, \bar{\lambda}, 1/\lambda, 1/\bar{\lambda}$. Moreover, every element of $\mathrm{Sp}(4, \mathbf{R})$ with such spectrum has this form. In particular, every such element is determined by the choice of λ with $|\lambda| > 1, \Im \lambda > 0$ and of the (ordered) decomposition $V = \mathbf{C}v \oplus \mathbf{C}w$ of V. Hence there is exactly one conjugacy class for each such λ. This proves (i).

To prove (ii), first observe that under the given hypotheses the eigenspace $V = E_\lambda$ has the same properties as in (i). Namely, $\beta|_V$ is a non-degenerate form of signature 0, and V is β-orthogonal to $\bar{V}$. Moreover, the restriction of A to V can be any β-invariant complex linear map with eigenvalue λ of multiplicity 2 and splitting number 0. Suppose that $A|_V$ has an eigenvector v with $\beta(v, v) \neq 0$. Then there is a β-orthogonal vector w such that $\beta(w, w) = -\beta(v, v)$, and because

$$\beta(w, w) = \beta(Aw, Aw) = \beta(\lambda w + \mu v, \lambda w + \mu v) = \beta(w, w) + |\mu|^2 \beta(v, v)$$

we must have $\mu = 0$. Thus A is diagonalizable (over $\mathbf{C}$) in this case. If no such v exists, we may choose a basis v, w for V such that $\beta(v, v) = \beta(w, w) = 0$ and so that $Av = \lambda v, Aw = \lambda w + \mu v$, for some $\mu \neq 0$. If we also set $\beta(v, w) = i$, it is easy to see that $\beta(Aw, Aw) = \beta(w, w)$ only if $\lambda \bar{\mu}$ is real. Given v, our choices have determined w up to a transformation of the form $w \mapsto w' = w + \kappa v$, where $\kappa \in \mathbf{R}$ since $\beta(w', w') = 0$. It is easy to check that if we make this alteration in w then μ does not change. On the other hand, if we rescale v, w replacing them by $v/\kappa, \bar{\kappa}w$ then μ changes by the positive factor $|\kappa|^2$. Hence we may assume that $\mu = \pm \lambda$. This shows that $A|_V$ is conjugate to exactly one of the matrices

$$D_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad N_\lambda^- = \begin{pmatrix} \lambda & 0 \\ -\lambda & \lambda \end{pmatrix}, \quad N_\lambda^+ = \begin{pmatrix} \lambda & 0 \\ \lambda & \lambda \end{pmatrix}.$$

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To prove (iii), observe that the eigenspace W of λ (where $|\lambda| > 1$) is Lagrangian, and by suitably conjugating A by an element of G we may suppose that the other eigenspace is JW. It then follows that A has the form $\lambda C \oplus \lambda^{-1} J^T (C^{-1})^T J$ for some $C \in \mathrm{SL}(2, \mathbf{R})$ with trace $= 2$. In fact, any linear map of the form $B \oplus J^T (B^{-1})^T J$, where $B \in \mathrm{GL}(2, \mathbf{R})$, is in G. It follows easily that there are two conjugacy classes, one in which C is diagonalizable, and one in which C is not.

(iv)   Finally, consider a matrix A in $\mathcal{B}_{\mathcal{U},\mathcal{R}}$ with a pair of eigenvalues $\lambda, \bar{\lambda}$, with $\lambda \in S^1 \cap \mathbf{H}$ where $\mathbf{H}$ is the upper half plane, and the eigenvalue say -1, with multiplicity 2. Assume say that the splitting number of λ is positive. Let V be the eigenspace generated by the eigenvectors v, w corresponding to the eigenvalues $\lambda, \bar{\lambda}$, and V' the invariant subspace associated to the double eigenvalue -1. By Lemma 2.5, v is β-orthogonal to V' and w, and $\beta(v, v) > 0$ by assumption. Similarly, w is β-orthogonal to V' and v, and $\beta(w, w) < 0$. Thus the restriction of β to V' is a also a non-degenerate symmetric Hermitian form with zero signature. Because A is non-diagonalizable, its restriction to V' is non-diagonalizable. Note that both eigenspaces V and V' are invariant under conjugation, and therefore that the restrictions of A to each subspace is real. This shows that we can identify $V \oplus V'$ with $\mathbf{C}(\mathbf{R}^2) \oplus \mathbf{C}(\mathbf{R}^2)$ by a linear map f which respects the factors, preserves β and is real, and such that $f A f^{-1}$ is the direct sum of a rotation on the first factor and of an element in $\mathcal{N}^\pm$ on the other factor.

The proof is similar in the other cases.   □

We next investigate the relationship between the positive cone field and the projection $\pi : \mathrm{G} \to \mathcal{C}onj$. Krein's Lemma 2.6 shows that the movement of eigenvalues along positive paths in $\mathcal{O}_{\mathcal{U}}$ and $\mathcal{O}_{\mathcal{U},\mathcal{R}}$ is constrained. However, this is not so for the other open strata.

Lemma 3.5 *The projection $\pi : \mathrm{G} \to \mathcal{C}onj$ maps the positive cone $\mathcal{P}_A$ at $A \in \mathcal{O}_{\mathcal{C}}$ onto the tangent space to $\mathcal{C}onj$ at $\pi(A)$. A similar statement is true for $A \in \mathcal{O}_{\mathcal{R}}$.*

Proof: Recall from Lemma 3.4 that an element A of $\mathcal{O}_{\mathcal{C}}$ defines a unique splitting $V \oplus \bar{V}$, where $\beta|_V$ has signature 0 and where $A|_V$ has eigenvalues $\lambda, 1/\bar{\lambda}$ for some $\lambda \in \{\Im z > 0\}$. Moreover, we may choose the eigenvectors v, w for $\lambda, 1/\bar{\lambda}$ respectively so that

$$\beta(v, v) = \beta(w, w) = 0, \quad \beta(v, w) = i.$$

Therefore, if we fix such v, w we get a unique representative of each conjugacy class by varying λ.

Most such splittings $V \oplus \bar{V}$ are not J-invariant, and so it is hard to describe the positive flows at A. However, because the positive cone field on G is invariant under conjugation (see Lemma 2.3), we only need to prove this statement for one representative of each conjugacy class, and so we choose one where this splitting

is J-invariant. For example, if we set $v = 1/\sqrt{2}(1, 0, i, 0)$ and $w = -Jv$, all the required identities are satisfied. Moreover, the transformation X which takes the standard basis to $v, w, \bar{v}, \bar{w}$ is unitary. Therefore, any linear automorphism B of V which is represented by a positive definite matrix with respect to the basis v, w extends by complex conjugation to a linear automorphism $B \oplus \bar{B}$ of $\mathbf{C}^4$ which is also positive definite with respect to the usual basis. This means that there are positive tangent vectors to G at A which preserve V and restrict to JB there. We are now essentially reduced to the 2-dimensional case. Just as in Lemma 3.1, if B is a real positive definite matrix with negative 21-entry then the eigenvalue λ moves along a ray towards S^1 and if its 21-entry is positive λ moves along this ray away from S^1. Since π maps the positive cone onto an open convex cone, this means that the image has to be the whole tangent space to $\mathcal{C}onj$.

The argument for $A \in \mathcal{O}_\mathcal{R}$ is similar but easier. $\qquad \square$

The next step is to study generic positive paths through the various parts of the codimension 1 boundary components. We will see that Krein's lemma forces the behavior of positive paths at the boundary $\mathcal{B}_\mathcal{U}$ to mimic that that near $-\mathbb{1}$ in the case $n = 1$, while the boundary $\mathcal{B}_\mathcal{R}$ is essentially not seen by positive paths. Our results may be summarized in the following proposition.

Proposition 3.6 (i) *If $A \in \mathcal{N}_\lambda^- \subset \mathcal{B}_\mathcal{U}$, then π maps the positive cone $\mathcal{P}_A$ into the set of vectors at $\pi(A)$ which point into $\pi(\mathcal{O}_\mathcal{C})$. Similarly, if $A \in \mathcal{N}_\lambda^+$, then $\pi_*(\mathcal{P}_A)$ consists of vectors pointing into $\pi(\mathcal{U})$.*

(ii) *If $A \in \mathcal{B}_\mathcal{R}$ then $\pi_*(\mathcal{P}_A)$ contains vectors which point into $\pi(\mathcal{O}_\mathcal{C})$ as well as vectors pointing into $\pi(\mathcal{O}_\mathcal{R})$.*

(iii) *If $A \in \mathcal{B}_{\mathcal{U},\mathcal{R}} \cup \mathcal{B}_{\mathcal{R},\mathcal{U}}$ then A is conjugate to an element which preserves the splitting $\mathbf{R}^4 = \mathbf{R}^2 \oplus \mathbf{R}^2$, and π_* takes the subcone in $\mathcal{P}_A$ formed by vectors which preserve this splitting onto the whole image $\pi_*(\mathcal{P}_A)$. In other words, positive paths through $\mathcal{B}_{\mathcal{U},\mathcal{R}}$ and $\mathcal{B}_{\mathcal{R},\mathcal{U}}$ behave just like paths in the product $\mathrm{SL}(2, \mathbf{R}) \times \mathrm{SL}(2, \mathbf{R})$.*

Proof: (i) Suppose first that $A \in \mathcal{N}_\lambda^-$. We claim that there is a neighbourhood of $A \in$ G whose intersection with $\mathcal{U}$ consists of elements which have an eigenvalue λ' with splitting number $+1$ which is close to λ and to the right of it, i.e with $\arg \lambda' < \arg \lambda$. To see this, observe first that by conjugacy invariance it suffices to prove this for a neighbourhood of A in the subgroup

$$\mathrm{G}_V = \{A' \in \mathrm{G} \;:\; A'(V) = V\}.$$

If $A' \in \mathrm{G}_V$, then $A'|_V$ may be written as $\lambda'Y$ where λ' is close to λ, and Y has determinant 1 and preserves β. If

$$Y = \left(\begin{array}{cc} a & b \\ c & d \end{array} \right)$$

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with respect to the basis v, w, then we must have $a\bar{c} \in \mathbf{R}$, $b\bar{d} \in \mathbf{R}$ and $a\bar{d} - c\bar{b} = 1$ in order to preserve β and $\mathrm{tr} Y = 2$. This implies that Y must be in $\mathrm{SL}(2, \mathbf{R})$. It is also close to the element $\frac{1}{\lambda} A|_V \in \mathcal{N}^- = \mathcal{N}_{-1}^-$. Therefore, the result follows from the corresponding result in the case $n = 1$: see the discussion just before Lemma 3.1.

Since eigenvalues with positive splitting number flow anticlockwise under a positive path (by Krein), every positive path through $A \in \mathcal{N}_\lambda^-$ must have all eigenvalues off S^1 for $t > T$ and so must enter $\mathcal{O}_\mathcal{C}$. Similarly, positive paths through any point of $\mathcal{N}_\lambda^+$ flow into $\mathcal{U}$. As in the case when $\lambda = -1$, this implies that positive paths through points of $\mathcal{D}_\lambda$ remain in $\mathcal{U}$.

(ii) Suppose first that A in the closure $\bar{\mathcal{B}}_\mathcal{R}$ of $\mathcal{B}_\mathcal{R}$ is diagonalizable, say $A = \mathrm{diag}(\lambda, 1/\lambda, \lambda, 1/\lambda)$. Then, as in Remark 3.2, there is a positive flow starting at A which keeps A diagonalizable and in $\mathcal{B}_\mathcal{D}$, while decreasing $|\lambda|$. Since any sufficiently C^1-small perturbation of this flow is still positive, we can clearly flow positively from A into both regions $\mathcal{O}_\mathcal{R}$ and $\mathcal{O}_\mathcal{C}$.

Next observe that if we fix $\lambda \in \mathbf{R}$ and consider only those elements of $\mathcal{B}_\mathcal{R}$ with fixed λ-eigenspace W and $1/\lambda$-eigenspace W', then these form a cone $C(\lambda)$ whose vertex is the diagonalizable element D_λ. (The structure here is just like that near $-\mathbb{1}$ which was discussed in §3.1.) Therefore, if v is a positive vector at D_λ which points into $\mathcal{O}_\mathcal{C}$, there is a nearby vector v' at a nearby (non-diagonalizable) point A which also points into $\mathcal{O}_\mathcal{C}$. Moreover, we may assume that v' is positive by the openness of the positive cone. Therefore one can move from A into $\mathcal{O}_\mathcal{C}$, and similarly into $\mathcal{O}_\mathcal{R}$. Since there is only one conjugacy class of such A for each λ, this completes the proof.

(iii) As shown in Lemma 3.1, the behavior of positive paths at the two conjugacy classes $-\mathbb{1}^\pm$ is dictated by Krein's lemma and the topology of $\mathrm{SL}(2, \mathbf{R})$. A similar argument applies here. $\qquad\square$

Remark 3.7 (i) One can construct explicit paths from $A \in \mathcal{B}_\mathcal{R}$ into $\mathcal{O}_\mathcal{C}$ and $\mathcal{O}_\mathcal{R}$ as follows. Note that the matrices $A \in \mathcal{B}_\mathcal{R}$ all satisfy the relation

$$\sigma_2 = \frac{\sigma_1^2}{4} + 2,$$

where σ_j denotes the jth symmetric function of the eigenvalues. Similarly, one can easily check that the matrices in $\mathcal{O}_\mathcal{C}$ satisfy the condition $\sigma_2 > \frac{\sigma_1^2}{4} + 2$ while those in $\mathcal{O}_\mathcal{R}$ satisfy $\sigma_2 < \frac{\sigma_1^2}{4} + 2$. Therefore, to see where the positive path $A_t = (1 + tJP + \ldots)A$ goes as t increases from 0 we just have to compare the derivatives

$$\sigma_2' = \frac{d}{dt}|_{t=0}\, \sigma_2, \quad \frac{\sigma_1 \sigma_1'}{2} = \frac{\sigma_1}{2} \frac{d}{dt}|_{t=0}\, \sigma_1.$$

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For example, if we take

$$A = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \frac{1}{\lambda} & 0 & \alpha \\ -\frac{\alpha}{\lambda^2} & 0 & \lambda & 0 \\ 0 & 0 & 0 & \frac{1}{\lambda} \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & -2 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

then

$$(1+tJPA) - \mu\mathbb{1} = \begin{pmatrix} \lambda - \mu - \frac{2\alpha t}{\lambda^2} & -\frac{5t}{\lambda} & 2t\lambda & -5\alpha t \\ t\lambda & \frac{1}{\lambda} - \mu & 0 & \alpha \\ -\frac{\alpha}{\lambda^2} & 0 & \lambda - \mu & -\frac{t}{\lambda} \\ -\frac{\alpha t}{\lambda^2} & -\frac{2t}{\lambda} & t\lambda & \frac{1}{\lambda} - \mu - 2\alpha t \end{pmatrix}.$$

Thus

$$\frac{\sigma_1 \sigma_1'}{2} = -2\alpha\left(\frac{2}{\lambda} + \lambda + \frac{1}{\lambda^3}\right)$$

while

$$\sigma_2' = -2\alpha\left(2\lambda + \frac{2}{\lambda^3}\right).$$

Therefore

$$\sigma_2' - \frac{\sigma_1 \sigma_1'}{2} = -2\alpha\left(\lambda + \frac{1}{\lambda^3} - \frac{2}{\lambda}\right) = -2\alpha(\lambda^{1/2} - \lambda^{-3/2})^2,$$

and the path goes into $\mathcal{O}_\mathcal{C}$ if $\alpha < 0$ and into $\mathcal{O}_\mathcal{R}$ if $\alpha > 0$.

(ii) Observe also that we did *not* show that π takes the positive cone at a point of $\mathcal{B}_\mathcal{R}$ onto the full tangent space of $\pi(\mathcal{O}_\mathcal{C}) = \mathbf{H}$. However, because of Lemma 2.1(iii), all that matters to us is that movement between these zones is possible.

4 Proof of the main results

4.1 The case $n \leq 2$

For the convenience of the reader we now restate Theorem 1.2.

Theorem 4.1 *Let $n \leq 2$.*

(i) *An element of $G - \mathcal{S}_1$ is the endpoint of a short positive path (from $\mathbb{1}$) if and only if it has an even number of real eigenvalues λ with $\lambda > 1$.*

(ii) *There is a positive path between any two elements $A, B \in G$. Moreover, any short positive path from $\mathbb{1}$ may be extended to a short positive path with endpoint in $\mathcal{U}$.*

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(iii) *The space of short positive paths (from $\mathbb{1}$) with endpoint in $\mathcal{U}$ is path-connected.*

Proof: (i) There are two ways eigenvalues can reach the positive real axis $\mathbf{R}^+$. Either a pair of eigenvalues moves from S^1 through $+1$ to a pair on $\mathbf{R}^+$, or a quadruplet moves from $\mathbf{C} - \mathbf{R}$ to $\mathbf{R}$. The first scenario cannot happen along a short path: for the only time a short path has an eigenvalue $+1$ is at time $t = 0$ and then, by Lemma 3.1, the eigenvalues have to move into S^1. In the second case, one gets an even number of eigenvalues on $(1, \infty)$.

(ii) Using Proposition 3.6 it is not hard to see that there is a positive path in $\mathcal{C}onj$ from $\pi(A)$ to $\mathbb{1}$ and from $\mathbb{1}$ to $\pi(B)$. The result now follows from Lemma 2.3. The second statement is proved similarly.

(iii) We have to show that any two short positive paths $\{A_t\}, \{B_t\}$ beginning at $\mathbb{1}$ and ending in $\mathcal{U}$ can be joined by a homotopy of short positive paths beginning at $\mathbb{1}$ and ending in $\mathcal{U}$. We may clearly assume that both paths are generic. (Note however that the homotopy of paths may have to go through codimension 2 strata.) Moreover, by Lemmas 2.1 and 2.6 we may assume that there is $\delta > 0$ such that $A_t = B_t \in \mathcal{U}$ for $t \in [0, \delta]$. Therefore, it will suffice to consider the case when the second path $\{B_t\}$ is (a reparametrization of) $A_t, 0 \leq t \leq \delta$. Thus we have to show how to "shrink" the path $\{A_t\}$ down to its initial segment in $\mathcal{U}$, keeping its endpoint in $\mathcal{U}$.

We do this by constructing a homotopy γ_t^μ in $\mathcal{C}onj$ between the projections of the two paths which at each time obeys the restrictions stated in Proposition 3.6. Indeed, such a homotopy means the existence of a homotopy $\tilde{\gamma}_t^\mu$ of short positive paths in G from $\mathbb{1}$ and ending in $\mathcal{U}$ which satisfies: $\tilde{\gamma}_t^0 = A_t$ for all t, $\tilde{\gamma}_0^\mu = \mathbb{1}$ and $\tilde{\gamma}_1^\mu \in \mathcal{U}$ for all μ, and $\pi(\tilde{\gamma}_t^1) = \pi(B_t)$. By Proposition 2.4, there is a homotopy of paths (with starting points equal to $\mathbb{1}$ and endpoints free in the conjugacy class of B_1) between $\{\tilde{\gamma}_t^1\}$ and $\{B_t\}$. The composition of these two homotopies is the desired path from $\{A_t\}$ to $\{B_t\}$.

We construct the homotopy γ^μ in $\mathcal{C}onj$ by means of a smooth map $r : \mu \mapsto r^\mu$ from the interval $[0, 1]$ to the space of short positive paths in $\mathcal{C}onj$ as follows.[4] Suppose that we have found r^μ which satisfy:
(i) $r^\mu(\mu) = \pi(A_\mu)$ and $r^\mu(1) \in \pi(\mathcal{U})$;
(ii) the paths r^μ are constant (with respect to t) for μ near 0.
Then it is not hard to check that we may take γ^μ to be:

$$\gamma^\mu(t) = \begin{array}{ll} \pi(A_t) & \text{if} \quad t \leq \mu \\ r^\mu(t) & \text{if} \quad t \geq \mu. \end{array}$$

We will construct the r^μ backwards, starting at $\mu = 1$. Intuitively, r^μ should be the simplest path from $\pi(A_\mu)$ to $\pi(\mathcal{U})$ (i.e. it should cross the stratum $\mathcal{C}onj_1$ as few times as possible), and we should think of the homotopy γ^μ as shrinking

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[4] As will become clear, we are only interested in $r^\mu(t)$ on the time interval $t \in [\mu, 1]$.

away the kinks in $\{A_t\}$ as μ decreases. To be more precise, let us define the **complexity** $c(\gamma)$ of a generic path $\gamma = \{C_t\}$ to be the number of times that $\pi(C_t)$ crosses $\mathcal{C}onj_1$. Then we will construct the r^μ so that $c(\gamma^\mu)$ does not increase as μ decreases. Moreover it decreases by 2 every time that A_μ moves (as μ decreases) from $\mathcal{O}_\mathcal{R}$ to $\mathcal{O}_\mathcal{C}$, or from $\mathcal{O}_{\mathcal{U},\mathcal{R}}$ to $\mathcal{O}_\mathcal{U}$ or from $\mathcal{O}_\mathcal{C}$ to $\mathcal{O}_\mathcal{U}$.

To this end we assume that, except near places where $\pi(A_\mu) \in \mathcal{C}onj_1$, the paths r^μ have the following form:

- if $A_\mu \in \mathcal{U}$ then r^μ is constant;
- if $A_\mu \in \mathcal{O}_\mathcal{C}$ is labelled with $\lambda = se^{i\theta}$ then r_μ goes down the ray $s'e^{i\theta}$ until it meets the circle in $\pi(\mathcal{N}^+_{e^{i\theta}})$ and enters $\pi(\mathcal{U})$;
- if $A_\mu \in \mathcal{O}_\mathcal{R}$ then r^μ moves the two eigenvalue pairs together, pushes them into $\pi(\mathcal{O}_\mathcal{C})$ and then follows the previous route to $\pi(\mathcal{U})$;
- if $A_\mu \in \mathcal{O}_{\mathcal{U},\mathcal{R}}$ then r^μ fixes the eigenvalues in S^1 and moves the real ones through $\pi(\mathcal{N}^+_{-1})$ to S^1.

We now describe how to extend r over each type of crossing. While doing this, there will be times μ at which we will want to splice different positive paths r^μ together. More precisely, at these μ_j we will have two different choices for the path r^{μ_j} from A_{μ_j} to $\mathcal{U}$. But these choices will be homotopic (by a homotopy which respects the endpoint conditions) and so, if we reparametrize the path A_t so that it stops at $t = \mu_j$ for a little while, we may homotop from one choice to the other. To be more precise, we choose a nondecreasing function $\rho : [0,1] \to [0,1]$ which is bijective over all points except the μ_j and is such that $\rho^{-1}(\mu_j)$ is an interval, and then change the relation between r^μ, γ^μ and A_t by requiring that

- $r^\mu(t)$ is defined for $t \in [\rho(\mu), 1]$ and $r^\mu(\rho(\mu)) = A_{\rho(\mu)}$,
- $\gamma^\mu(t) = A_t, t \le \rho(\mu)$.

First let us consider how to handle crossings of the stratum $\mathcal{B}_\mathcal{U}$. Such crossings take place either at $\mathcal{N}^+_\lambda$ or $\mathcal{N}^-_\lambda$. However, because positive paths starting at $\mathcal{N}^+_\lambda$ point into $\mathcal{U}$ there is no problem extending r smoothly over this type of crossing. Observe that at this crossing the complexity remains unchanged. The problem comes when $A_\mu \in \mathcal{N}^-_\lambda$. To deal with this, first let C_t be a positive path which starts in $\mathcal{U}$, crosses $\mathcal{N}^-_\lambda$ at time μ, goes a little into $\mathcal{O}_\mathcal{C}$ and then crosses back into $\mathcal{U}$ through $\mathcal{N}^+_\lambda$. We may choose this path so that it is homotopic through positive paths with fixed endpoints to a positive path lying entirely in $\mathcal{U}$. (In fact, we could start with a suitable path in $\mathcal{U}$ and then perturb it.) By Lemma 2.1, we may suppose that $\pi(A_t) = \pi(C_t)$ for t in some interval $(\mu - \varepsilon, \mu + \varepsilon)$ and so (using the splicing technique described above) we may choose r so that, for some $\mu' \in (\mu, \mu + \varepsilon)$, we have $r^{\mu'}(t) = \pi(C_t)$ for $t \ge \mu'$. We now use this formula to extend r over the interval $(\mu - \varepsilon, \mu')$. Then $r^{\mu-\varepsilon} = \{C_t\}$ starts and ends in $\mathcal{U}$ and goes only a little way outside $\mathcal{U}$. We now reparametrize $\{A_t\}$ stopping it at time $t = \mu - \varepsilon$ so that there is time first to homotop $r^{\mu-\varepsilon} = \{C_t\}$ with fixed endpoints to a path in $\mathcal{U}$ and then to shrink it,

fixing its first endpoint $A_{\mu-\varepsilon}$, to a constant path. This completes the crossing. Note that the complexity does decrease by 2.

A little thought shows that the same technique may be used to deal with crossings of the other boundaries $\mathcal{B}_{\mathcal{U},\mathcal{R}}, \mathcal{B}_{\mathcal{R},\mathcal{U}}$ and $\mathcal{B}_{\mathcal{R}}$. For example, if as μ decreases A_μ passes from $\mathcal{O}_\mathcal{C}$ to $\mathcal{O}_\mathcal{R}$, one easily extends r but does not change the complexity of γ^μ. On the other hand, if A_μ passes from $\mathcal{O}_\mathcal{R}$ to $\mathcal{O}_\mathcal{C}$ one needs to take more trouble in extending r but in exchange one decreases the complexity. □

4.2 The case $n > 2$

As previously explained, the eigenvalues of an element of $\mathrm{Sp}(2n, \mathbf{R})$ form complex quadruplets or pairs on S^1 or $\mathbf{R}$. It is easy to check that the only singularities (or bifurcations) encountered by a generic 1-dimensional path $\gamma = A_{t\in[0,1]}$ are those in which

(a) a pair of eigenvalues on S^1 or $\mathbf{R}$ become equal to ± 1;

(b) two pairs of eigenvalues coincide on S^1 or $\mathbf{R}$ and then move into $\mathbf{C}$ (or conversely).

In particular, two quadruplets or three pairs do not coincide generically. Therefore, for each t, the space $\mathbf{R}^{2n}$ decomposes into a sum of 2- and 4-dimensional eigenspaces $E_1(t) \oplus \ldots \oplus E_k(t)$. Moreover, the interval $[0, 1]$ may be divided into a finite number of subintervals over which this decomposition varies smoothly with t. (The type of the decomposition may be different in the different pieces.) We claim that within each such piece the eigenvalue flow is just the same as it would be if the decomposition were fixed rather than varying. The reason for this is that all the restrictions on the eigenvalue flow are forced by Krein's lemma, which is valid for arbitrary variation of decomposition, together with topological data concerning Conj (i.e. topological information on the way the types of conjugacy classes fit together.) With these remarks it is not hard to adapt all the above arguments to the general case. In particular, the proof of (iii) is not essentially more difficult when $n > 2$ since in this case we reduce the complexity of the path as we procede with the homotopy. It is important to note that there still are essentially unique ways of choosing the paths r from the different components of the top stratum $\mathcal{C}_0$ into $\pi(\mathcal{U})$. ("Essentially unique" means that the set of choices is connected.) For example, if A^μ were on a stratum with 3 eigenvalue pairs on $\mathbf{R}^-$, then one might combine two to make a quadruplet in $\mathbf{C}$ which then moves down a ray to S^1 leaving one pair to move through $\mathcal{N}_{-1}^+$ to S^1, or one might move all 3 pairs directly down $\mathbf{R}$ to S^1. But these paths are homotopic (through positive paths with endpoints on $\pi(\mathcal{U})$) and so it is immaterial which choice we make. □

5 Positive paths in Hamiltonian systems and in Hofer's geometry

We present here a brief outline of the various ways in which positive paths intervene in the stability theory of Hamiltonian systems, and in Hofer's geometry. Actually, they are also a crucial ingredient of the theory of closed geodesics as developed by Bott in a series of papers ([3, 4, 5]). But since the application of positive paths to closed geodesics (and in particular to the computation of the index of iterates of a given closed geodesic) can also be presented within the framework of positive Hamiltonian systems via the geodesic flow (where Ekeland's formula appears as a generalization of Bott's iteration formula), this application is, at least theoretically, reducible to the following one.

5.1 Periodic Hamiltonian systems

Many Hamiltonian systems are given as periodic perturbations of autonomous flows. When both the autonomous Hamiltonian and the perturbation are quadratic maps $\mathcal{Q}_t$, the fundamental solution of the periodic system

$$
\begin{aligned}
\dot{x}(t) &= -J\nabla_X \mathcal{Q}_t(x(t)), \quad \text{where} \quad \mathcal{Q}_t = \mathcal{Q}_{t+1}, \\
x(0) &= \xi
\end{aligned}
$$

is a path of matrices $A_t \in \mathrm{G}$ as in §1. (In other words, the trajectory $x(t)$ which starts at ξ is $A_t(\xi)$.) By the periodicity of the generating Hamiltonian, it is clear that $A_{k+t} = A_t A_1^k$ for all integers k and real numbers $t \in [0, 1)$.

Definition 5.1 The above periodic system is **stable** if all solutions $x(t)$ remain bounded for all times (in other words, it is stable if there is a constant C such that $\|A_t\| \leq C$ for all $t > 0$). A matrix A is **stable** if there is C such that $\|A^k\| \leq C$ for all positive integers k.

Clearly, the above system is stable if and only if $A = A_1$ is stable. But it is easily seen that a matrix A is stable if and only if all its eigenvalues lie on the unit circle. Thus the stability of the periodic Hamiltonian system is determined by the spectrum of its time-1 flow A.

Definition 5.2 The Hamiltonian system is **strongly stable** if any C^2-small periodic (and quadratic) perturbation of it remains stable. A matrix A is **strongly stable** if it has a neighbourhood which consists only of stable matrices.

Since a C^2-small perturbation $\mathcal{Q}'_t$ of the Hamiltonian $\mathcal{Q}_t$ leads to a flow $A'_{t\in[0,1]}$ which is C^1-close (as a path) to the unperturbed flow, the time-1 map $A' = A'(1)$ is C^0-close to the time-1 map $A = A(1)$. Conversely any matrix

A' close enough to A in the symplectic group is the time-1 map of a C^2-small quadratic perturbation of $\mathcal{Q}_{t \in [0,1]}$. This means that a periodic Hamiltonian system is strongly stable if and only if the time-1 map A is strongly stable. Hence such a system is strongly stable exactly when its time-1 flow belongs to the *interior* of the set of symplectic matrices with spectrum in S^1. But we have seen in the previous sections that this interior consists of all matrices with spectrum in $S^1 - \{\pm 1\}$ and maximal splitting numbers. (More precisely, at each multiple eigenvalue the absolute value of the splitting number must equal the multiplicity of the eigenvalue.) This is the Stability Theorem due to Krein and Gelfand-Lidskii.

Now consider the case $n = 2$ and assume that the stable periodic Hamiltonian $\mathcal{Q}_t$ is negative and that its flow $\{A_t\}$ is short (that is the index of $\{A_t\}$ is zero). Further, let $\mathcal{NU}$ be the union of all *open* strata for which none of the eigenvalues lie on S^1, and let $e_A(s)$ be the number of times that the path $\{A_t\}$ enters $\mathcal{NU}$ and comes back to $\mathcal{O}_\mathcal{U}$ during the time interval $(0, s]$.

Lemma 5.3 *For every short positive path $A_{t \in [0,1]}$, beginning at $\mathbb{1}$, $e_A(1) \leq 1$.*

Proof: Because $\mathcal{NU}$ and $\mathcal{U}$ are open, it suffices to prove this when the path $\{A_t\}$ is generic. If $e_A(1) = 0$ there is nothing to prove. So suppose that $e_A(s) = 1$ for some $s \leq 1$. Then Proposition 3.6 implies that on the interval $[0, s]$ either A_t moves from $\mathcal{O}_\mathcal{C}$ to $\mathcal{O}_\mathcal{U}$ passing through some conjugacy class $\mathcal{N}_\lambda^+$ at some time $t_0 < s$ or A_t moves from $\mathcal{O}_\mathcal{R}^-$ into $\mathcal{O}_\mathcal{U}$ via matrices with all eigenvalues on $(-\infty, -1) \cup S^1$. In either case, it follows from Krein's lemma 2.6 that the first eigenvalue of A_s that one encounters when traversing S^1 anticlockwise from $\mathbb{1}$ has negative splitting number. Therefore, by Lemma 2.7, the eigenvalues of $A_t, t \geq s$, can only leave S^1 after a pair has crossed $+1$. $\qquad\square$

More generally, this argument shows that for a stable positive path $A_{t \in [0,1]}$, which is not necessarily short or generic, $e_A(1) \leq i + 1$, where $i = i_A(1)$ is the Conley-Zehnder index of $A_{t \in [0,1]}$. (In this context, $i_A(1)$ is defined to be the number of times $t > 0$ that a generic positive perturbation $\{A_t'\}$ of $\{A_t\}$ (with $A_0' = A_0 = \mathbb{1}$) crosses the set $\mathcal{S}_1 = \{A : \det(A - \mathbb{1}) = 0\}$. Observe that because $\{A_t'\}$ is positive each such crossing occurs with positive orientation.)

Combining this with Ekeland's version of Bott's iterated index formula (see [8]), we get:

Proposition 5.4 *Let $A_{t \geq 0}$ be a stable positive path generated by a 1-periodic Hamiltonian in $\mathbf{R}^4$. Then there is a positive real number a such that for any time $t > 0$:*

$$e_A(t) \leq a \, [t] \, i_A(1) + 1$$

where $[t]$ is the greatest integer less than or equal to t.

Thus some aspects of the qualitative behavior of such paths are linearly controlled by the Conley-Zehnder index. Similarly, the number of times that

such a path A_t leaves $\mathcal{O}_\mathcal{U}$, enters any other open stratum and comes back to $\mathcal{O}_\mathcal{U}$ during the time interval $(0, t]$, is bounded above by

$$3(i_A(t) + 1) \leq 3(a\,[t]\,i_A(1) + 1).$$

As a last application, noted by Krein in [10], Chap VI, consider the system $\dot{x}(t) = -\mu J \nabla_X \mathcal{Q}_t(x(t))$, $\mathcal{Q}_{t+1} = \mathcal{Q}_t$, with a parameter $\mu \in \mathbf{R}$. Then:

Proposition 5.5 (Krein) *If each $\mathcal{Q}_t$ is negative definite, there is $\mu_0 > 0$ such that the periodic Hamiltonian system*

$$\dot{x}(t) = -\mu J \nabla_X \mathcal{Q}_t(x(t))$$

is strongly stable for all $0 < \mu < \mu_0$.

Actually, the value μ_0 is the smallest μ such that the time-1 flow A_μ of the system has at least one eigenvalue equal to -1. Hence each time-1 flow A_μ for $\mu < \mu_0$ must be strongly stable.

5.2   Hofer's geometry

Hofer's geometry is the geometry of the group $\mathrm{Ham}(M)$ of all smooth Hamiltonian diffeomorphisms, generated by Hamiltonians $H_{t\in[0,1]}$ with compact support, of a given symplectic manifold M, endowed with the bi-invariant norm

$$\|\phi\| = \inf_{H_t} \int_0^1 \left(\max_M H_t - \min_M H_t \right) dt$$

where the infimum is taken over all smooth compactly supported Hamiltonians $H_{t\in[0,1]}$ on M whose time-1 flows are equal to ϕ. This norm defines a Finslerian metric (which is L^∞ with respect to space and L^1 with respect to time) on the infinite dimensional Fréchet manifold $\mathrm{Ham}(M)$. In [15] and [2], it is shown that a path ψ_t generated by a Hamiltonian H_t is a geodesic in this geometry exactly when there are two points $p, P \in M$ such that p is a minimum of H_t and P is a maximum of H_t for all t. We then showed in [16] that the path $\psi_{t\in[0,1]}$ is a stable geodesic (that is a local minimum of the length functional) if the linearized flows at p and P have Conley-Zehnder index equal to 0. (See also [20].) Note that the linear flow at p is positive while the one at P is negative. The condition on the Conley-Zehnder index means that both flows are short. The proof of the above sufficient condition for the stability of geodesics in [16] relies on the application of holomorphic methods which reduce the problem to the purely topological Main Theorem 1.2. Hence the result of the present paper can be considered as the topological part of the proof of the stability criterion in Hofer geometry. We refer the reader to [16] for the equivalence between this stability criterion and the local squeezability of compact sets in cylinders.

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